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DIFFUSION, CONVECTION AND CHEMICAL REACTION IN A CHANNEL

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Abstract—For laminar flow between a flat conduit exact solutions of the problem of steady-state diffusion and convection are presented for constant wall concentration, constant flux across the wall and a heterogeneous chemical reaction of the first order. The initial conditions were considered to be arbitrary functions of the coordinate across the channel. The same was performed for the cases of simultaneous diffusion, convection and homogeneous chemical reaction for constant wall concentration and heterogeneous chemical reaction. The eigenvalues are presented in tables and the eigenfunctions are presented in analytical form, which contains the confluent hypergeometric function.

NOMENCLATURE

b,	width of channel;
c,	concentration, c_w wall concentration, c_i initial concentration, c_0 concentration outside the duct, $\bar{c}$ mean concentration across the channel;
D,	diffusion coefficient;
Da,	$=\dfrac{kb}{2D}$, Damköhler number;
f_0,	constant flux across the channel wall;
${}_1F_1$,	confluent hypergeometric function;
k,	heterogeneous chemical reaction coefficient;
k^*,	homogeneous chemical reaction coefficient;
p,	pressure of liquid;
Pe,	$= ReSc$, Péclet number;
Re,	$=\dfrac{2u_0 b}{3\nu}$, Reynolds number;
Sc,	$=\dfrac{\nu}{D}$, Schmidt number;
u_0,	maximum flow velocity in the channel (at $y = 0$);
x,	axial coordinate;
y,	coordinate perpendicular to channel axis;
y^*,	$= y\Big/\dfrac{b}{2}$.
β_n,	eigenvalues;
Γ,	gamma function;
γ,	incomplete gamma function;
η,	dynamic viscosity of liquid;
κ^2,	proportionality factor;
Ψ,	Ψ-function.

1. INTRODUCTION

THE PROBLEM of steady state diffusion and convection in a straight channel has been treated previously for certain boundary conditions. Mathematically it represents nothing but the problem of forced heat convection in laminar flow assuming the Poiseuille flow velocity profile [1–3]. During the course of the analysis the partial differential equation has been transformed into an ordinary differential equation describing the concentration in the direction normal to the channel. The solution of this ordinary differential equation and the eigenvalues have been obtained numerically. Only the lower eigenvalues were given, until less than twenty years ago [4] asymptotic values were obtained for higher modes.

The purpose of this paper is to present the exact solution of the problem of steady state diffusion and convection with various boundary conditions, including the problem of a heterogeneous chemical reaction of the first order, as well as the problem of simultaneous diffusion, convection and chemical reaction. All eigenvalues may be easily and expediently obtained from the confluent hypergeometric function. In addition the orthogonality relations for the various cases are presented. With these the determination of the remaining integration constants may be performed even for arbitrarily given initial conditions exhibiting a function of the normal coordinate as their initial concentration distribution at the tube inlet.

2. BASIC EQUATIONS AND SOLUTION

For the determination of the local concentration of a component in a moving liquid between two parallel plates with a fully developed velocity profile, where molecular diffusion in axial direction is neglected, the second order differential equation (Fig. 1)

$$\frac{\partial^2 c}{\partial y^2}-\frac{u_0}{D}\left[1-\frac{y^2}{(b/2)^2}\right]\frac{\partial c}{\partial x}-\frac{k^*}{D}c=0 \tag{1}$$

has to be solved. The width of the channel is b and

$$u_0=-\frac{\partial p}{\partial x}\frac{b^2}{8\eta}$$

is the maximum velocity of the liquid. With the substitution $y^*=y/(b/2)$ this yields

$$\frac{\partial^2 c}{\partial y^{*2}}-\frac{u_0 b^2}{4D}\left[1-y^{*2}\right]\frac{\partial c}{\partial x}-\frac{k^* b^2}{4D}c=0. \tag{1'}$$

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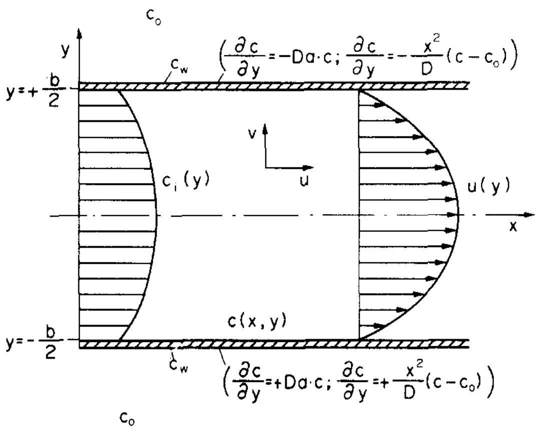

FIG. 1. Geometry of the system.

This problem encounters the absorption of a substance by another through which it can diffuse convectively and molecularly and with which it can react chemically. It is assumed here, that the diffusing substance is immobilized by an irreversible first order reaction. If there is no simultaneous homogeneous chemical reaction the homogeneous chemical reaction coefficient k^* vanishes.

The problem has to be solved with the boundary condition

$$\mp A \frac{\partial c}{\partial y^*} = Bc + C \quad \text{at} \quad y^* = \pm 1 \tag{2}$$

which contains the cases of

(a) constant surface concentration, i.e. $A = 0$, $B = 1$, $C = -c_w$
(b) heterogeneous chemical reaction of the first order, i.e. $A = D$, $B = kb/2$, $C = 0$
(c) evaporation, i.e. $A = D$, $B = \kappa^2 b/2$, $C = -(\kappa^2 b/2)\cdot c_0$
(d) constant flux across the wall, i.e. $A = D$, $B = 0$, $C = f_0 b/2$.

If a chemical reaction takes place in the surface of the channel wall, we talk then about a heterogeneous chemical reaction, in addition to the homogeneous chemical reaction. The value $kb/2D = Da$ is called the Damköhler number. Assuming

$$c = c_1(y^*) + c_2(x, y^*)$$

the partial differential equation (1′) yields

$$\frac{d^2 c_1}{dy^{*2}} - \frac{k^* b^2}{4D} c_1 = 0 \tag{3}$$

with

$$\mp A \frac{\partial c_1}{\partial y^*} = Bc_1 + C \quad \text{at} \quad y^* = \pm 1$$

and

$$\frac{\partial^2 c_2}{\partial y^{*2}} - \frac{u_0 b^2}{4D}(1 - y^{*2})\frac{\partial c_2}{\partial x} - \frac{k^* b^2}{4D} c_2 = 0 \tag{4}$$

with

$$\mp A \frac{\partial c_2}{\partial y^*} = Bc_2 \quad \text{at} \quad y^* = \pm 1.$$

The solution of equation (3) yields

$$c_1 = -\frac{C\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}} y^*\right]}{\left\{B\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right] + A\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\sinh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right]\right\}}. \tag{5}$$

With $c_2 = \bar{C}e^{-\alpha x}$ equation (4) exhibits with $\beta^2 \equiv (u_0 b^2 \alpha/4D)$ [5] the solution

$$\bar{C} = \bar{A}\, e^{-\frac{\beta}{2}y^{*2}}\, {}_1F_1\left(\tfrac{1}{4}\left(1 - \beta + \frac{k^* b^2}{4D\beta}\right), \tfrac{1}{2}; \beta y^{*2}\right) \tag{6}$$

where ${}_1F_1$ represents the confluent hypergeometric series

$${}_1F_1(\alpha, \gamma; z) = \sum_{\lambda=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\alpha+\lambda)z^\lambda}{\Gamma(\alpha)\Gamma(\gamma+\lambda)\lambda!}.$$

The boundary condition for c_2 yields the equation for the determination of the eigenvalues β_n:

$$B = \frac{A}{2}\left(1 + \beta + \frac{k^* b^2}{4D\beta}\right) - \frac{A}{2}\left(1 - \beta + \frac{k^* b^2}{4D\beta}\right) \times \frac{{}_1F_1\left(\frac{1}{4}\left(5 - \beta + \frac{k^* b^2}{4D\beta}\right), \frac{1}{2}; \beta\right)}{{}_1F_1\left(\frac{1}{4}\left(1 - \beta + \frac{k^* b^2}{4D\beta}\right), \frac{1}{2}; \beta\right)}. \tag{7}$$

These eigenvalues β_n are presented for constant surface concentration, for heterogeneous chemical reaction and evaporation, and for constant flux ($k^* = 0$-case) in Tables 1–3. The solution of the problem is then given by the expression

$$c = -\frac{C\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}} y^*\right]}{\left\{B\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right] + A\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\sinh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right]\right\}} + \sum_{n=1}^{\infty} \bar{A}_n e^{-\frac{\beta_n}{2}y^{*2}} \cdot {}_1F_1\left(\tfrac{1}{4}\left(1 - \beta_n + \frac{k^* b^2}{4D\beta_n}\right), \tfrac{1}{2}; \beta_n y^{*2}\right) \times e^{-\frac{4\beta_n^2 D}{u_0 b^2}x} \tag{8}$$

The constants $\bar{A}_n$ are obtained from the initial condition at the inlet $x = 0$, which reads $c(0, y^*) = c_i(y^*)$ and results for c_2 in

$$c_2(0, y^*) = c_i(y^*) + \frac{C\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}} y^*\right]}{\left\{B\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right] + A\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\sinh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right]\right\}}. \tag{9}$$

With the abbreviation

$$\bar{C}_n(y^*) \equiv e^{-\frac{\beta_n}{2}y^{*2}} \cdot {}_1F_1\left(\tfrac{1}{4}\left(1 - \beta_n + \frac{k^* b^2}{4D\beta_n}\right), \tfrac{1}{2}; \beta_n y^{*2}\right) \tag{10}$$

which satisfies the differential equation

$$\frac{d^2\bar{C}}{dy^{*2}} + \left[\beta^2(1 - y^{*2}) - \frac{k^* b^2}{4D}\right]\bar{C} = 0$$

Table 1. β_n-values for constant wall concentration

	$\frac{k^*b^2}{4D}$				
n	0	0.1	1.0	10	100
1	1.680706	1.705408	1.973668	3.753472	10.517979
2	5.668724	5.677812	5.777087	6.804868	12.807847
3	9.667103	9.667270	9.731214	10.417786	15.480656
4	13.666522	13.667353	13.711603	14.217085	18.474449
5	17.666236	17.667500	17.700241	18.100188	21.716535
6	21.666069	21.668438	21.692764	22.026755	25.147685
7	25.665961	25.669741	25.687459	25.974962	28.701792
8	29.665886	29.669244	29.683496	29.936296	32.364750
9	33.665832	33.668869	33.680420	33.906250	36.085944
10	37.665791	37.668575	37.677963	37.882206	39.851775

Table 2(a). β_n-values for various Damköhler-numbers and $\frac{k^*b^2}{4D}$-values

	Da				
n	0.1	1.0	10	100	
1	0.374941	1.000000	1.550312	1.666035	
2	4.332076	4.654545	5.397629	5.636888	
3	8.327079	8.542446	9.293999	9.613329	
4	12.328510	12.498990	13.217682	13.603544	
5	16.328308	16.472657	17.157799	17.592962	
6	20.328448	20.454484	21.108539	21.583684	
7	24.328478	24.440997	25.066797	25.574979	
8	28.328517	28.430519	29.030707	29.566814	
9	32.328542	32.422099	32.999038	33.559075	$\frac{k^*b^2}{4D} = 0$
10	36.328563	36.415158	36.970926	37.551700	
11	40.328580	40.409318	40.945737	41.544639	
12	44.328594	44.404324	44.922990	45.537853	
13	48.328605	48.399996	48.902310	49.531312	
14	52.328615	52.396200	52.883399	53.524989	
15	56.328624	56.392840	56.866021	57.518866	
16	60.328631	60.389841	60.849977	61.512923	
17	64.328637	64.387143	64.835107	65.507147	
18	68.328643	68.384702	68.821127	69.501523	

Table 2(b). β_n-values for various Damköhler-numbers and $\frac{k^*b^2}{4D}$-values

	Da				
n	0.1	1.0	10	100	
1	0.535742	1.062995	1.588564	1.701239	
2	4.354768	4.675074	5.411420	5.649568	
3	8.327215	8.556917	9.308817	9.627288	
4	12.320443	12.516351	13.234488	13.615060	
5	16.318533	16.488570	17.172599	17.603504	$\frac{k^*b^2}{4D} = 0.1$
6	20.317606	20.469858	21.121587	21.593502	
7	24.317032	24.455803	25.078521	25.584289	
8	28.316629	28.444917	29.041443	29.575743	
9	32.316329	32.436162	33.009016	33.567708	
10	36.316096	36.428949	36.980305	37.560097	
1	1.260792	1.524110	1.891290	1.979940	
2	4.555467	4.851578	5.536855	5.763207	
3	8.437837	8.661265	9.375170	9.687909	
4	12.408832	12.576553	13.277000	13.650642	
5	16.391302	16.533770	17.206441	17.627249	$\frac{k^*b^2}{4D} = 1.0$
6	20.380700	20.504780	21.150324	21.609643	
7	24.373283	24.483994	25.103766	25.595218	
8	28.367904	28.468167	29.064093	29.582809	
9	32.363802	32.455677	33.029653	33.571799	
10	36.360577	36.445530	36.999332	37.561820	

Table 2(c). β_n-values for various Damköhler-numbers and $\frac{k^*b^2}{4D}$-values

	Da				
n	0.1	1.0	10	100	
1	3.634360	3.662817	3.728017	3.751998	
2	6.164637	6.302725	6.650157	6.786674	
3	9.458632	9.623018	10.144622	10.382907	
4	13.123793	13.268780	13.853489	14.169056	
5	16.920771	17.071151	17.670843	18.041197	$\frac{k^*b^2}{4D}=10$
6	20.808017	20.947287	21.544618	21.957940	
7	24.733057	24.850416	25.447850	25.897039	
8	28.678850	28.779554	29.364876	29.849860	
9	32.637726	32.726025	33.295308	33.811783	
10	36.605433	36.684193	37.236902	37.780114	
1	10.512867	10.512884	10.512967	10.513046	
2	12.803990	12.804609	12.807572	12.810292	
3	15.440815	15.444667	15.462683	15.478601	
4	18.349319	18.361220	18.416132	18.463462	
5	21.475687	21.499196	21.607042	21.698263	$\frac{k^*b^2}{4D}=100$
6	24.758169	24.796645	24.976323	25.124777	
7	28.191912	28.239946	28.467537	28.662403	
8	31.719559	31.776408	32.056028	32.308009	
9	35.345068	35.407249	35.726379	36.029106	
10	39.045408	39.109924	39.455769	39.790099	

we obtain the orthogonality relation

$$\int_0^1 (1-y^{*2})\bar{C}_m(y^*)\bar{C}_n(y^*)\,dy^*$$

$$=\begin{cases} 0 \text{ for } m\neq n \\ \dfrac{1}{2\beta_n}\left[\dfrac{\partial\bar{C}_n}{\partial\beta_n}\cdot\dfrac{\partial\bar{C}_n}{\partial y^*}\right]_{y^*=1} \quad \text{for constant wall concentration} \\ \dfrac{1}{2\beta_n}\left[\dfrac{\partial\bar{C}_n}{\partial y^*}\dfrac{\partial\bar{C}_n}{\partial\beta_n}-\bar{C}_n\dfrac{\partial^2\bar{C}_n}{\partial y^*\partial\beta_n}\right]_{y^*=1} \quad \text{all other cases} \\ \quad = -\dfrac{\bar{C}_n(1)}{2\beta_n}\left[\dfrac{\partial^2\bar{C}_n}{\partial y^*\partial\beta_n}+\dfrac{B}{A}\dfrac{\partial\bar{C}_n}{\partial y^*}\right]_{y^*=1} \quad \text{for } m=n. \end{cases}$$

The value for $m=n$ was obtained from an indeterminate expression by applying the rule of l'Hospital. The values

$$\left[\frac{\partial\bar{C}_n}{\partial\beta_n}\right]_{y^*=1} \quad \text{and} \quad \left[\frac{\partial\bar{C}_n}{\partial y^*}\right]_{y^*=1}$$

may be obtained from (10) and the confluent hypergeometric function. They are given by [6]:

Table 3. β_n-values for constant flux across the walls ($k^*=0$)

n	β_n
1	4.286418
2	8.303518
3	12.291126
4	16.297552
5	20.299898
6	24.301932
7	28.303424
8	32.304607
9	36.305567
10	40.306364
11	44.307038
12	48.307617
13	52.308121
14	56.308564
15	60.308957
16	64.309308
17	68.309624
18	72.309911
19	76.310172
20	80.310410

$$\left[\frac{\partial\bar{C}_n}{\partial y^*}\right]_{y^*=1}=\tfrac{1}{2}e^{-\frac{\beta_n}{2}}\left[1-\beta_n+\frac{k^*b^2}{4D\beta_n}\right]{}_1F_1\left(\tfrac{1}{4}\left(5-\beta_n+\frac{k^*b^2}{4D\beta_n}\right),\tfrac{1}{2};\beta_n\right)$$
$$-\tfrac{1}{2}e^{-\frac{\beta_n}{2}}\left[1+\beta_n+\frac{k^*b^2}{4D\beta_n}\right]{}_1F_1\left(\tfrac{1}{4}\left(1-\beta_n+\frac{k^*b^2}{4D\beta_n}\right),\tfrac{1}{2};\beta_n\right).$$

and

$$\left[\frac{\partial\bar{C}_n}{\partial\beta_n}\right]_{y^*=1}=\frac{e^{-\frac{\beta_n}{2}}}{4\beta_n}\,{}_1F_1\left(\tfrac{1}{4}\left(5-\beta_n+\frac{k^*b^2}{4D\beta_n}\right),\tfrac{1}{2};\beta_n\right)\left[1-\beta_n+\frac{k^*b^2}{4D\beta_n}\right]$$

$$-\frac{e^{-\frac{\beta_n}{2}}}{4}\left[1+\frac{1}{\beta_n}+\frac{k^*b^2}{4D\beta_n}\right]{}_1F_1\times\left(\tfrac{1}{4}\left(1-\beta_n+\frac{k^*b^2}{4D\beta_n},\tfrac{1}{2};\beta_n\right)\right)-\tfrac{1}{4}\left(1+\frac{k^*b^2}{4D\beta_n^2}\right)\sum_{\lambda=0}^{\infty}\frac{2^{2\lambda}\beta_n^{\lambda}}{(2\lambda)!}\frac{\Gamma\left(\lambda+\tfrac{1}{4}\left(1-\beta_n+\frac{k^*b^2}{4D\beta_n}\right)\right)}{\Gamma\left(\tfrac{1}{4}\left(1-\beta_n+\frac{k^*b^2}{4D\beta_n}\right)\right)}$$

$$\times\left[\Psi\left(\lambda+\tfrac{1}{4}\left(1-\beta_n+\frac{k^*b^2}{4D\beta_n}\right)\right)-\Psi\left(\tfrac{1}{4}\left(1-\beta_n+\frac{k^*b^2}{4D\beta_n}\right)\right)\right]e^{-\beta_n/2}. \quad (11)$$

From the initial condition (9) the constants $\bar{A}_n$ may be determined by the expression

$$\bar{A}_n=\frac{2\beta_n\int_0^1(1-y^{*2})\left[c_i(y^*)+\dfrac{C\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}y^*\right]}{\left\{B\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right]+A\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\sinh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right]\right\}}\right]\bar{C}_n(y^*)\,dy^*}{\left[\dfrac{\partial\bar{C}_n}{\partial y^*}\dfrac{\partial\bar{C}_n}{\partial\beta_n}-\delta\bar{C}_n\dfrac{\partial^2\bar{C}_n}{\partial y^*\partial\beta_n}\right]_{y^*=1}} \quad (12)$$

$$\delta=\begin{cases}0 & \text{for constant wall concentration}\\ 1 & \text{for all other cases.}\end{cases}$$

If c_i is constant the first part of the integral may be evaluated and yields

$$\int_0^1(1-y^{*2})\bar{C}_n(y^*)\,dy^*=\frac{k^*b^2}{4D\beta_n^2}\int_0^1\bar{C}_n(y^*)\,dy^*-\frac{\bar{C}_n'(1)}{\beta_n^2}. \quad (13)$$

For the determination of the remaining integral and that of equation (12) we refer to the appendix.

If at the inlet $x=0$ the initial condition c_i is a constant value, the integration constants are with (11) given by

$$\bar{A}_n=-\frac{2[c_i-c_w]}{\beta_n\left[\dfrac{\partial\bar{C}_n}{\partial\beta_n}\right]_{y^*=1}}$$

for constant wall concentration and $k^*=0$

$$\bar{A}_n=-\frac{2c_i}{\beta_n\left[\dfrac{\partial\bar{C}_n}{\partial\beta_n}+\dfrac{1}{Da}\dfrac{\partial^2\bar{C}_n}{\partial y^*\partial\beta_n}\right]_{y^*=1}}$$

for heterogeneous chemical reaction and $k^*=0$.

3. CONSTANT FLUX ACROSS THE WALL

The case of just diffusion and convection with a constant rate of diffusing material across the boundary

$$\left(\frac{\partial c}{\partial y^*}=\mp\frac{f_0b}{2D}\quad\text{at}\quad y^*=\pm1\right)$$

needs special treatment. For the solution of the differential equation (1′) with $k^*=0$, we assume $c=c_1+c_2$, where c_1 and c_2 have to satisfy (1′) with the boundary condition

$$\frac{\partial c_1}{\partial y^*}=\mp\frac{f_0b}{2D}\quad\text{and}\quad\frac{\partial c_2}{\partial y^*}=0\quad\text{for}\quad y^*=\pm1 \quad (14)$$

respectively. With

$$c_1=Bx+f^*(y^*)$$

and c_i = constant we obtain

$$c_1=c_i-\frac{3f_0}{bu_0}x-\frac{3f_0b}{8D}[y^{*2}-\tfrac{1}{6}y^{*4}-\tfrac{13}{70}] \quad (15)$$

The concentration c_2 is given by the series expression with $k^*=0$:

$$c_2(x,y^*)=\sum_{n=1}^{\infty}\bar{A}_n e^{-\frac{\beta_n}{2}y^{*2}}\cdot{}_1F_1(\tfrac{1}{4}(1-\beta_n),\tfrac{1}{2};\beta_ny^{*2})\times e^{-\frac{4\beta_n^2D}{u_0b^2}\cdot x}. \quad (16)$$

The integration constants $\bar{A}_n$ are obtained from the initial condition, with the orthogonality relation ($\bar{C}_n'(1)=0$)

$$\int_0^1(1-y^{*2})\bar{C}_m(y^*)\bar{C}_n(y^*)\,dy^*=\begin{cases}0 & \text{for } m\neq n\\ -\dfrac{\bar{C}_n(1)}{2\beta_n}\cdot\dfrac{\partial^2\bar{C}_n}{\partial y^*\partial\beta_n}(1) & \text{for } m=n.\end{cases} \quad (17)$$

Thus $\bar{A}_n$ is with the initial condition for the concentration c_2 at the inlet $x=0$

$$c_2(0,y^*)=\frac{3f_0b}{8D}[y^{*2}-\tfrac{1}{6}y^{*4}-\tfrac{13}{70}] \quad (18)$$

obtained from

$$\bar{A}_n=-\frac{3f_0\beta_nb\int_0^1(1-y^{*2})[y^{*2}-\tfrac{1}{6}y^{*4}-\tfrac{13}{70}]\bar{C}_n(y^*)\,dy^*}{4D\bar{C}_n(1)\cdot\dfrac{\partial^2\bar{C}_n}{\partial y^*\partial\beta_n}(1)} \quad (19)$$

which yields

$$\bar{A}_n=-\frac{f_0b}{D\beta_n\dfrac{\partial^2\bar{C}_n}{\partial y^*\partial\beta_n}(1)}.$$

4. REMARKS

In the previous treatment we have neglected the molecular diffusion in axial direction. This, however, is not always permitted. Without this neglection the process would be described by the differential equation

$$\frac{\partial^2 c}{\partial y^{*2}}+\frac{b^2}{4}\frac{\partial^2 c}{\partial x^2}-\frac{u_0 b^2}{4D}(1-y^{*2})\frac{\partial c}{\partial x}-\frac{k^* b^2}{4D}c=0.$$

In the usual way the substitution $c=\bar{C}\,e^{-\alpha x}$ shall reduce the above equation to the ordinary differential equation

$$\frac{d^2\bar{C}}{dy^{*2}}+\left[\beta^2(1-y^{*2})+\frac{4D^2\beta^4}{b^2u_0^2}-\frac{k^*b^2}{4D}\right]\bar{C}=0.$$

The solution of this equation is

$$\bar{C}=e^{-\frac{\beta}{2}y^{*2}}\,{}_1F_1\left(\tfrac{1}{4}\left(1-\beta-\frac{4D^2\beta^3}{b^2u_0^2}+\frac{k^*b^2}{4D\beta}\right),\tfrac{1}{2};\beta y^{*2}\right).$$

which yields

$$\frac{\partial^2 c}{\partial \eta^{*2}}-\frac{u_0 b^2}{2D}\eta^*\frac{\partial c}{\partial x}=0$$

in the case of diffusion and convection, and

$$\frac{\partial^2 c}{\partial \eta^{*2}}-\frac{u_0 b^2}{2D}\eta^*\frac{\partial c}{\partial x}-\frac{k^* b^2}{4D}c=0$$

in the case of simultaneous diffusion, convection and chemical reaction. By proper transformation these differential equations may be solved, as indicated by Levich [7] for the first equation. For the solution of the second differential equation a Laplace-Transform-technique may be applied.

Table 4. β_n-values for various Péclet-numbers (for constant wall concentration)

	Pe				
n	1	10	100	1000	∞
1	0.977152	1.636176	1.680206	1.680702	1.680706
2	1.828636	4.574453	5.645687	5.668488	5.668724
3	2.387047	6.563249	9.549618	9.665858	9.667103
4	2.838052	8.120181	13.337954	13.662894	13.666523
5	3.227349	9.436836	16.979712	17.658252	17.666236
6	3.571120	10.599105	20.458014	21.651175	21.666607
7	3.887497	11.647333	23.770739	25.641056	25.665961
8	4.178400	12.612735	26.922088	29.627370	29.665886
9	4.452324	13.509006	29.921406	33.609669	33.665832
10	4.713051	14.351016	32.780426	37.586289	37.665791

From this one may conclude immediately, that the above solutions tend into those of the previous section, if

$$\frac{4D^2\beta^2}{b^2u_0^2}\ll 1$$

is satisfied. This means that the Péclet number $Pe=Re\,Sc\gg 1$. Only for such cases the molecular diffusion in axial direction may be neglected. It may, however, be mentioned that the functions are not orthogonal.

To show just the difference of the eigenvalues β_n for the classical Graetz-problem, the equation for the determination of the eigenvalues β_n with $Re=(2u_0 b/3\nu)$

$${}_1F_1\left(\tfrac{1}{4}\left(1-\beta-\frac{16\beta^3}{9(Pe)^2}\right),\tfrac{1}{2};\beta\right)=0$$

has been evaluated for various Péclet numbers (Table 4).

It may also be noted that the representation of the results in the immediate vicinity of the inlet $x=0$ requires a large number of eigenvalues and eigenfunctions. This could be achieved with some increased numerical effort with the above solution. Another method could as well be adopted by noting that near $x=0$ the concentration change takes place very closely to the wall in the extremely thin diffusion layer. This suggests immediately the transformation of the governing differential equation to the wall by

$$\eta=\frac{b}{2}-y \quad \text{or} \quad \eta^*=1-y^*$$

5. MEAN CONCENTRATION AND CONCENTRATION AT THE WALL

The concentration at the walls may be easily obtained from the previous results by merely introducing in the solutions presented in Sections 2 and 3 the value $y^*=\pm 1 (y=\pm b/2)$. For constant flux across the walls the wall-concentration yields

$$c_w=c_i-\frac{3f_0}{bu_0}x-\frac{17f_0 b}{70D}+\sum_{n=1}^{\infty}\bar{A}_n e^{-\frac{\beta_n}{2}}\cdot{}_1F_1(\tfrac{1}{4}(1-\beta_n),\tfrac{1}{2};\beta_n)\cdot e^{-\frac{4\beta_n^2 D}{u_0 b^2}x} \quad (20)$$

For simultaneous diffusion, convection and chemical reaction one finds at the walls

$$c_w=-\frac{C}{\left\{B+A\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\tanh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right]\right\}}+\sum_{n=1}^{\infty}\bar{A}_n e^{-\frac{\beta_n}{2}}\times{}_1F_1\left(\tfrac{1}{4}\left(1-\beta_n+\frac{k^*b^2}{4D\beta_n}\right),\tfrac{1}{2};\beta_n\right)\cdot e^{-\frac{4\beta_n^2 D}{u_0 b^2}x} \quad (21)$$

where the values A, B, C have to be introduced appropriately for the various cases.

The mean concentration over the cross section of the channel is given by ($\bar{u}=\frac{2}{3}u_0$)

$$\bar{c}=\frac{2}{b\bar{u}}\int_0^{b/2}u(y)c(x,y)\,dy=\tfrac{3}{2}\int_0^1(1-y^{*2})c(x,y^*)\,dy^* \quad (22)$$

which is only a function of the axial coordinate x. Introducing the previous results and applying the appropriate boundary conditions and orthogonality relations yields:

$$\bar{c} = c_i - \frac{3f_0}{bu_0}x \quad \text{for constant flux across the walls and } k^* = 0 \quad (23)$$

and

$$\bar{c} = \frac{12CD\left[1-\frac{2}{b}\left(\frac{D}{k^*}\right)^{\frac{1}{2}}\tanh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right]\right]}{k^*b^2\left\{B+A\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\tanh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}\right]\right\}}$$

$$+\frac{3}{2}\sum_{n=1}^{\infty}\frac{\bar{A}_n}{\beta_n^2}e^{-\frac{4D\beta_n^2}{u_0b^2}x}\cdot\left\{\frac{k^*b^2}{4D}\int_0^1 \bar{C}_n(y^*)\,dy^*+\frac{B}{A}\bar{C}_n(1)\right\} \quad (24)$$

for all other cases.

The integral $\int_0^1 \bar{C}_n(y^*)\,dy^*$ is presented in the Appendix.

REFERENCES

1. L. Graetz, Über die Wärmeleitungsfähigkeit von Flüssigkeiten, *Ann. Phys.* **18**, 79–94 (1883); **25**, 337–357 (1885).
2. W. Nusselt, Abhängigkeit der Wärmeübergangszahl von der Rohrlänge, *Z. Ver. D. Ing.* **54**, 1154–1158 (1910).
3. M. Jakob, *Heat Transfer*, Vol. 1. John Wiley, New York (1949).
4. J. R. Sellars, M. Tribus and J. S. Klein, Heat transfer to laminar flow in a round tube or flat conduit—the Graetz problem extended, *Trans. Am. Soc. Mech. Engrs* **78**, 441 (1956).
5. J. Pahor and K. Strand, A note on heat transfer in laminar flow through a gap, *Appl. Scient. Res.* **A10**, 81–84 (1961).
6. E. Jahnke, F. Emde and F. Lösch, *Tafeln höherer Funktionen*. B. G. Teubner, Stuttgart (1960).
7. V. G. Levich, *Physicochemical Hydrodynamics*. Prentice Hall, Englewood Cliffs, NJ (1962).

APPENDIX

For the determination of the integration constants of the previous sections various integrals appear, which have to be evaluated. They are of the form

$$J = \int_0^1 f(y^*)C_n(y^*)\,dy^*$$

where $f(y^*) = 1;\ y^{*2m};\ y^{*2m+1}$ or combinations of those. It is therefore [6]

$$J_1 = \int_0^1 \bar{C}_n(y^*)\,dy^*$$

$$\int_0^1 e^{-\frac{\beta_n}{2}y^{*2}}\sum_{\lambda=1}^{\infty}\frac{\Gamma(\alpha_n+\lambda)2^{2\lambda}\beta_n^{\lambda}}{\Gamma(\alpha_n)(2\lambda)!}y^{*2\lambda}\,dy^*$$

which may be reduced with the transformation

$$\beta_n y^{*2} = z$$

to

$$J_1 = \sum_{\lambda=0}^{\infty}\frac{\Gamma(\alpha_n+\lambda)2^{2\lambda-1}}{\Gamma(\alpha_n)(2\lambda)!\sqrt{\beta_n}}\int_0^{\beta_n}e^{-z/2}z^{\lambda-1/2}\,dz.$$

The integral is presented by

$$\int_0^{\beta_n}e^{-z/2}z^{\lambda-1/2}\,dz = 2^{\lambda+1/2}\gamma(\lambda+\tfrac{1}{2},\tfrac{1}{2}\beta_n)$$

where γ represents the incomplete gamma-function, which may be represented in series form as [6]

$$\gamma(\lambda+\tfrac{1}{2},\tfrac{1}{2}\beta_n) = \sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\beta_n^{\nu+\lambda+1/2}}{\nu!2^{\nu+\lambda+1/2}(\nu+\lambda+\frac{1}{2})}.$$

The integral is therefore given by

$$J_1 = \sum_{\lambda=0}^{\infty}\frac{\Gamma(\alpha_n+\lambda)2^{2\lambda-1}}{\Gamma(\alpha_n)(2\lambda)!}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\beta_n^{\nu+\lambda}}{2^{\nu}\nu!(\nu+\lambda+\frac{1}{2})}.$$

The following integrals may be obtained in a similar way. They are given by the expressions:

$$J_2 = \int_0^1 y^{*2m}C_n(y^*)\,dy^*$$

$$\sum_{\lambda=0}^{\infty}\frac{\Gamma(\alpha_n+\lambda)2^{2\lambda-1}}{\Gamma(\alpha_n)(2\lambda)!}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\beta_n^{\nu+\lambda}}{\nu!2^{\nu}(\lambda+\nu+m+\frac{1}{2})}$$

$$J_3 = \int_0^1 y^{*2m+1}C_n(y^*)\,dy^*$$

$$= \sum_{\lambda=0}^{\infty}\frac{\Gamma(\alpha_n+\lambda)2^{2\lambda-1}}{\Gamma(\alpha_n)(2\lambda)!}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\beta_n^{\nu+\lambda}}{2^{\nu}\nu!(\nu+\lambda+m+1)}$$

$$= \sum_{\lambda=0}^{\infty}\frac{\Gamma(\alpha_n+\lambda)2^{3\lambda+m}(m+\lambda)!}{\Gamma(\alpha_n)(2\lambda)!\beta_n^{m+1}}\cdot\left\{1-e^{-\beta_n/2}\sum_{\nu=0}^{m+\lambda}\frac{\beta_n^{\nu}}{\nu!2^{\nu}}\right\}$$

All appearing integrals in the text may be determined with these results. It is furthermore

$$\int_0^1(1-y^{*2})\cosh\left[\frac{b}{2}\left(\frac{k^*}{D}\right)^{\frac{1}{2}}y^*\right]C_n(y^*)\,dy^*$$

$$= \sum_{m=0}^{\infty}\frac{\left(\frac{k^*b^2}{4D}\right)^m}{(2m)!}\cdot\left\{\sum_{\lambda=0}^{\infty}\frac{\Gamma(\alpha_n+\lambda)2^{2\lambda-1}}{\Gamma(\alpha_n)(2\lambda)!}\times\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\beta_n^{\nu+\lambda}}{2^{\nu}\nu!(\nu+\lambda+m+\frac{1}{2})(\nu+m+\lambda+\frac{3}{2})}\right\}.$$

DIFFUSION, CONVECTION ET REACTION CHIMIQUE DANS UN CANAL

Résumé—On présente des solutions exactes du problème de diffusion et de convection stationnaires en écoulement laminaire dans un canal plat, pour une concentration constante à la paroi, un flux pariétal constant et une réaction chimique hétérogène du premier ordre. Les conditions initiales sont supposées fonctions arbitraires de la coordonnée transversale dans le canal. On a traité de même les cas d'une diffusion et convection avec réaction chimique homogène pour une concentration de paroi constante et une réaction chimique hétérogène. Les valeurs propres sont présentées dans des tables et les fonctions propres sont données sous forme analytique, à l'aide de la fonction hypergéométrique confluente.

DIFFUSION, KONVEKTION UND CHEMISCHE REAKTION IN EINEM KANAL

Zusammenfassung—Für laminare Strömung in einem flachen Kanal wurden exakte Lösungen des Problems der stationären Diffusion und Konvektion angegeben bei konstanter Wandkonzentration, konstantem Wärmestrom an der Wand und heterogener chemischer Reaktion von erster Ordnung. Die Anfangsbedingungen wurden als beliebige Funktionen der Kanalkoordinaten angesehen.

Dasselbe Vorgehen erfolgte für den Fall gleichzeitiger Diffusion, Konvektion und homogener chemischer Reaktion bei konstanter Wandkonzentration und heterogener chemischer Reaktion. Eigenwerte wurden in Tabellen angegeben und Eigenfunktionen in analytischer Form, die die konfluente hypergeometrische Funktion enthielten.

ДИФФУЗИЯ, КОНВЕКЦИЯ И ХИМИЧЕСКАЯ РЕАКЦИЯ В КАНАЛЕ

Аннотация — Получены точные решения задачи установившейся диффузии и конвекции для граничных условий постоянной концентрации на стенке, постоянного потока и гетерогенной химической реакции первого порядка при ламинарном течении в плоском канале. Предполагается, что начальные условия являются произвольными функциями поперечной координаты. Эти же условия принимаются для случаев совместной диффузии, конвекции и гомогенной химической реакции при постоянной концентрации на стенке и гетерогенной химической реакции. Собственные значения затабулированы, а собственные функции представлены аналитически в виде сходящейся гипергеометрической функции.